ENGINEERING ANALYSIS

we begin our studying ordinary differential equations (ODEs) by deriving them from physical or other problems **(modeling),** solving them by standard methods, and interpreting solutions in terms of a given problem. We begin with the simplest ODEs, called ODEs ***of the first order*** because they involve only the first derivative of the unknown function, no higher derivatives. Our usual notation for the unknown function will be $y(x)$, or $y(t)$ if the independent variable is time t.

Modeling

If we want to solve an engineering problem (usually of physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions, equations . Such an expression is known as a mathematical **model** of the given problem. The process of setting up a model, solving it mathematically, and interpreting the result in physical or other terms is called *mathematical modeling* or, briefly, **modeling.** We shall illustrate this process by FOLLOWING examples and problems.

An **ordinary differential equation (ODE)** is an equation that contains one or several derivatives of an unknown function, which we usually call $y(x)$ (or sometimes $y(t)$ if the independent variable is time t). The equation may also contain y itself, known functions of x (or t), and constants. For example,

(a) $y' = \cos x$,

(b) $y'' + 9y = 0$,

An ODE is said to be of order n if the nth derivative of the unknown function y is the highest derivative of y in the equation. The concept of order gives a useful classification into ODEs of first order, second order, and so on. Thus, (a) is of first order, (b) of second.

1- Application "first order differential equation"

1- ذوبان الاملاح في المحاليل

ان المعادلة العامة المستخدمة لحساب كمية الملح داخل الحوض

$$\frac{dw}{dt} = Q_{in}C_{in} - Q_{out}C_{out}$$

حيث ان w : كمية الملح داخل الحوض في فترة زمنية t

Q_{in} : التصريف الداخل الى الحوض = $\frac{volume}{time}$

C_{in}: تركيز الملح للمحلول الداخل الى الحوض= $\frac{mass}{volume}$

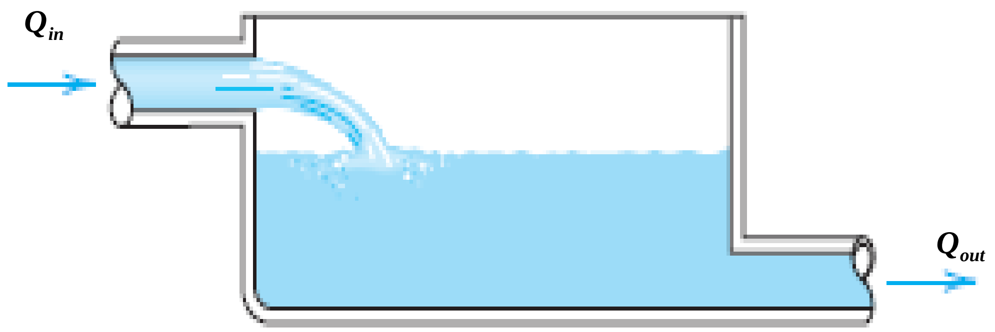
C_{out}: تركيز الملح للمحلول الخارج من الحوض

$Qout$: التصريف الخارج من الحوض

$$C_{out} = \frac{w}{v + (Q_{in} - Q_{out})t}$$

Example 1

The tank in Fig. 1 contains 1000 gal of water in which initially 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min, and each gallon contains 5 lb of dissolved salt. The mixture in the tank is kept uniform by stirring. Brine runs out at 10 gal/min. Find the amount of salt in the tank at any time *t*.



Solution. Step 1. Setting up a model. Let $w(t)$ denote the amount of salt in the tank at time *t*.

$$C_0 = \frac{w}{v} = \frac{100}{1000} = 0.1\ lb/\ gal \qquad \text{Initial condition}$$

$$C_{in} = \frac{w}{v} = \frac{5}{1000} = 0.005\ lb/\ gal$$

$$\frac{dw}{dt} = Q_{in}C_{in} - Q_{out}C_{out}$$

$$\frac{dw}{dt} = 10*0.005 - 10*\frac{w}{1000} = \frac{50 - 10w}{1000}$$

$$\frac{dw}{dt} = \frac{50 - 10w}{1000}$$

Step 2. Solution of the model. is separable. Separation, integration, and taking exponents on both sides give:

$$\frac{dw}{dt} = \frac{50 - 10w}{1000}$$

$$\frac{dw}{50 - 10w} = \frac{dt}{1000}$$

Integrate both sides we get:

$$\int \frac{dw}{50 - 10w} = \int \frac{dt}{1000}$$

$$\ln (50 - 10w) = 0.001\, t + c$$

$$50 - w = e^{0.001t+c} = k\, e^{0.001\, t}$$

$$w = 50 - k\, e^{0.001\, t} \quad \text{------------------------(1)}$$

To find the constant k use the boundary condition:

At time t = 0 , $C_0 = 0.1 \; lb/gal$ **.**

$\therefore\ w_o = 100 \; lb$ **Substitute in equation 1**

$$50 - 100 = K$$

$$\therefore k = -50$$

$\therefore$ **General equation is :**

$$w = 50 + 50\, e^{0.001\, t} = 50 \; (1 + e^{0.001\, t})$$

The amount of salt at any time is:

$$\boxed{w = 50\,(1 + e^{0.001\, t})}$$

Leaking Tank. Outflow of Water Through a Hole (Torricelli's Law):

اذا كان لديك خزان مملوء بالماء وكان الخزان يحتوي على ثقب فان الماء يبدأ بالتسرب خلال الزمن وبمرور الزمن عمق الماء سوف يبدأ بالنقصان لذلك قام تورشلي بوضع معادلة يمكن من خلالها حساب مقدار ارتفاع الماء داخل الخزان وفي ازمان مختلفة.
المعادلة العامة لتورشلي هي كالاتي :

$$A(y)\frac{dy}{dt} = Q_{in} - Q_{out}$$

حيث ان : $A(y)$ **تمثل مساحة الشريحة عند عمق y**

التصريف الداخل الى الحوض Q_{in} :

كمية الملح الخارج من الحوض في فترة t : *Qout*

$$Q_{out} = cd * \pi * r^2 \sqrt{2 g y}$$

cd : معامل التخصر ويعتمد على مادة الخزان ويؤخذ عادة =

r : نصف قطر الفتحة التي يتسرب منها الماء

g : تعجيل ارضي

y : بعد الشريحة عن الفتحة التي يتسرب منها الماء

EXAMPLE 2

This is another prototype engineering problem that leads to an ODE. It concerns the outflow of water from a cylindrical tank with a hole at the bottom (Fig. 2). A cylindrical tank with the radius of 2 m, the hole has diameter 2 cm, and the initial height of the water when the hole is opened is 4 m. Find a)Depth of water at any time b) The time for the water needed to reach (1m) c) The time to empty the tank?

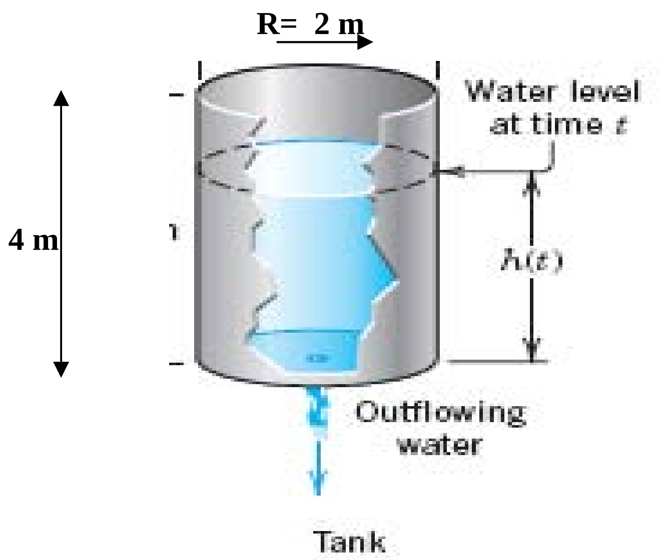


Fig. 2

Solution:

$$A(y)\frac{dy}{dt} = Q_{in} - Q_{out}$$

$$A(y) = \pi(2)^2 = 4\pi \quad m^2$$

$$Q_{out} = cd * \pi * r^2 \sqrt{2\,g\,y}$$

$$4\pi\frac{dy}{dt} = 0 - 1*\pi*\left(\frac{1}{100}\right)^2\sqrt{2*9.8*y}$$

$$4\frac{dy}{\sqrt{y}} = 0 - 4.427*10^{-4}dt$$

$$\int \frac{4}{4.427*10^{-4}}\frac{dy}{\sqrt{y}} = \int -dt$$

$$\int 9035.46\; y^{-0.5}dy = \int -dt$$

$$\frac{9035.46\; y^{0.5}}{0.5} = -t + c$$

$$18070.9\sqrt{y} = -t + c$$

لايجاد قيمة الثابت c يجب استخدام ال **boundary condition**

y= H = 4 m **at t = 0**

$$18070.9\sqrt{4} = -0 + c$$

C = 36141.857

$$18070.9\sqrt{y} = -t + 36141.857$$

لايجاد الزمن الذي يصبح فيه عمق الماء داخل الخزان واحد متر

$$18070.9\sqrt{1} = -t + 36141.857 \quad \Rightarrow \quad t = 18070.957 \text{ sec}$$

To find the time required to empty the tank is :

$$18070.9\sqrt{0} = -t + 36141.857 \quad \Rightarrow \quad t = 36141.857 \text{ sec}$$

SECOND ORDER DIFFERENTIAL EQUATION WITH VARIABLE COEFFICIEWNTS

لهذا النوع من المعادلات هناك طريقتين للحل . طريقة خاصة وطريقة عامة وكما هو مبين ادناه:
الطريقة الخاصة: ويتم استخدامها مع المعادلات التفاضلية من الدرجة الثانية ذات المعاملات المتغيرة با استعمال (Cauchy equation or Euller equation

وللحل بهذه الطريقة يجب توفر الشروط التالية
يجب ان تكون درجة الاسس للمعامل بنفس درجة المشتقة المضروب بها

$$Ax^2y'' + Bxy' + cy = 0$$

طريقة الحل :

1- نفرض $y = x^m$ حيث ان m مجهولة وهو ثابت ثم نجد المشتقات المطلوبة وتعويضها في المعادلة المعطاة وتقسم الناتج على x^m .

2- المعادلة الناتجة تكون بدلالة m ثم نجد قيمتها.

ا- في حالة جذور مختلفة وحقيقية $m_1 \neq m_2$

$$y_h = c_1 x^{m_1} + c_2 x^{m_2}$$

ب- في حالة الجذور مختلفة وخيالية

$$m_{1,2} = \alpha \mp \beta i$$

$$y_h = x^{\alpha}(c_1 \sin \beta \ln x + c_2 \cos \beta \ln x)$$

ج- في حالة الجذور الحقيقية متطابقة

$$m_1 = m_2 = m$$

$$y_h = (c_1 + c_2 \ln x)\, x^m$$

ملاحظة : في بعض الاحيان يتحقق الشرط اعلاه حول الاساس (x-a) بدلا من x في هذه الحالة نتبع نفس الخطوات اعلاه وبدلا من x نضع (x-a):

$$y_h = c_1(x-a)^{m_1} + c_2 (x-a)^{m_2}$$

Example:

Solve second order differential equation

$$\left(x^2 - 2x + 1) \; y'' + 5(x-1) \; y' + 6y = 0 \right)$$

Solution:

Let :

$y = (x-1)^m$

$y' = m(x-1)^{m-1}$

$y'' = m(m-1)(m-2)^{m-2}$

Substitute in the original equation

$x^2 - 2x + 1) \; y'' + 5(x-1) \; y' + 6y = 0$

$(x-1)^2 y'' + 5(x-1)y' + 6y = 0$

$m(m-1)(x-1)^2(x-1)^{m-2} + 5(x-1)(m)(x-1)^{m-1} + 6(x-1)^m = 0$

$m(m-1)(x-1)^m + 5(m)(x-1)^m + 6(x-1)^m = 0$

نقسم المعادلة على $(x-1)^m$

$m(m-1) + 5m + 6 = 0$

$m^2 - m + 5m + 6 = 0$

$m^2 + 4m + 6 = 0$

$$m_{1,2} = \frac{-4 \mp \sqrt{16 - 4*1*6}}{2}$$

$m_{1,2} = -2 \mp \sqrt{2}i$

$y_h = (x-1)^2 (c_1 \sin\sqrt{2}\ln(x-1) + c_2 \cos\sqrt{2}\ln(x-1))$

ثانيا الطريقة العامة :

SOLUTION OF SECOND ORDER DIFFERENTIAL EQUATION BY

<u>FROBENIUS METHOD (POWER SERIES)</u>

Power Series Solutions

ان الصيغة العامة للمعادلات التفاضلية من الدرجة الثانية التي تحل بهذه الطريقة

$$F_1(x)\,y'' + F_2(x)\,y' + F_3(x)\,y = 0$$

1-نفرض الاتي

$$y = \sum_{n=0}^{n=\alpha} an \quad x^{n+r}$$

2- نجد كل من

$$y' = \sum an(n+r)x^{n+r-1}$$

$$y'' = \sum an(n+r)(n+r-1)\ x^{n+r-2}$$

ويتم تعويض كل من $y\ ,\ \ y'\ ,\ y''$ في المعادلة التفاضلية من الدرجة الثانية

3- ناخذ معاملات اصغر اس بعد التعويض ونعوض بقيمة n= 0 للتخلص من $\sum$

نحصل على جذرين $r_1\ ,\ r_2$

هناك ثلاث حالات :

Case 1

If $r_1 \neq r_2 \rightarrow r_1 - r_2$ *not* integer

Case 2

If $r_1 = r_2$

Case 3

If $r_1 \neq r_2 \rightarrow r_1 - r_2$ *Integer*

4-يتم ايجاد المتوالية y_1

$$y_1 = \sum_{n=0}^{n=\alpha} anx^{n+r_1}$$

5- يتم ايجاد المتوالية y_2 بالاعتماد على الحالات الثلاثة المذكورة سابقا

Case 1-

$$y_2 = \sum_{n=0}^{n=\alpha} b_n\, x^{n+r_2}$$

Case 2-

$$y_2 = \Phi y_1 \quad If \ y_1 \ is \ familiar$$

$$y_2 = A\, y_1 \ln x + \sum b_n x^{n+r_2} \quad If \ y_1 \ is \ not \ familiar$$

$$\Phi = \int \frac{e^{-\int p(x)dx}}{y_1^2} \qquad p(x) = \frac{f_2(x)}{f_1(x)}$$

Case 3

$$y_2 = \sum_{n=0}^{n=\alpha} b_n x^{n+r_2} \qquad \text{if } b_n \text{ can be found as } a_n$$

$$y_2 = \Phi y_1 \quad If \ y_1 \ is \ familiar$$

$$y_2 = A\, y_1 \ln x + \sum b_n x^{n+r_2} \quad If \ y_1 \ is \ not \ familiar$$

Example :

Solve the following second order differential equation by Frobenius (power series) :-

$$6x^2y'' + 7xy' - (1 + x^2)y = 0, \qquad (0 < x < \infty)$$

$$6x^2 \sum_0^\infty (n+r)(n+r-1)a_n x^{n+r-2} + 7x \sum_0^\infty (n+r)a_n x^{n+r-1}$$
$$-(1+x^2)\sum_0^\infty a_n x^{n+r} = 0$$

$$\sum_0^\infty 6(n+r)(n+r-1)a_n x^{n+r} + \sum_0^\infty 7(n+r)a_n x^{n+r}$$
$$-\sum_0^\infty a_n x^{n+r} - \sum_0^\infty a_n x^{n+r+2} = 0.$$

$$\sum_0^\infty \left\{\left[6(n+r)^2 + n + r - 1\right] a_n - a_{n-2}\right\} x^{n+r} = 0,$$

$$\left[6(n+r)^2 + n + r - 1\right] a_n - a_{n-2} = 0$$

for each $n = 0, 1, 2, \ldots$.

In particular, $n = 0$ gives

$$\left(6r^2 + r - 1\right) a_0 - a_{-2} = 0$$

or, since $a_{-2} \equiv 0$,

$$\left(6r^2 + r - 1\right) a_0 = 0.$$

Proceeding, with $a_0 \neq 0$, it follows

$$6r^2 + r - 1 = 0$$

so $r = -1/2$ and $1/3$.

First, set $r = -1/2$.

$$
\begin{aligned}
n = 1: &\quad a_1 = a_{-1} = 0, \\
n = 2: &\quad a_2 = \frac{1}{14} a_0, \\
n = 3: &\quad a_3 = \frac{1}{39} a_1 = 0, \\
n = 4: &\quad a_4 = \frac{1}{76} a_2 = \frac{1}{(76)(14)} a_0, \\
n = 5: &\quad a_5 = \frac{1}{125} a_3 = 0, \\
n = 6: &\quad a_6 = \frac{1}{186} a_4 = \frac{1}{(186)(76)(14)} a_0,
\end{aligned}
$$

and so on. From these results we have the solution

$$
\begin{aligned}
y(x) &= a_0 x^{-1/2} \left[1 + \frac{1}{14} x^2 + \frac{1}{(76)(14)} x^4 + \frac{1}{(186)(76)(14)} x^6 + \cdots \right] \\
&= a_0 y_1(x),
\end{aligned}
$$

Next, set $r = 1/3$.

$$
a_n = \frac{1}{6\left(n + \frac{1}{3}\right)^2 + n - \frac{2}{3}} a_{n-2},
$$

$$
\begin{aligned}
y(x) &= a_0 x^{1/3} \left[1 + \frac{1}{34} x^2 + \frac{1}{(116)(34)} x^4 + \frac{1}{(246)(116)(34)} x^6 + \cdots \right] \\
&= a_0 y_2(x),
\end{aligned}
$$

SYSTEM OF SIMULTANEOUS LINEAR DIFFERENTIAL EQUATION

المعادلات التفاضلية الانية : وهي المعادلات التي تحوي على اكثر من متغير تابع (بسط المشتقة) ومتغير مستقل (مقام المشتقة) وان طريقة الحل با اتباع المثال التالي:-

Example:

Solve the following differential equations:

$$2\frac{dx}{dt} + \frac{dy}{dt} - 4x - y = e^t \quad --------1$$

$$\frac{dx}{dt} + 3x + \frac{d^2y}{dt^2} + y = 0 ---------2$$

Solution:

اولا نحول المشتقة الى صيغة D

$$D = \frac{d}{dt} \quad \Rightarrow \quad D^2 = \frac{d^2}{dt^2}$$

نعوض في المعادلات المعطاة (1 و 2) وكالاتي :

$$2Dx + Dy - 4x - y = e^t \quad --------3$$

$$Dx + 3x + D^2y + y = 0 ---------4$$

ثانيا : نرتب المعادلة بحيث كل متغير تابع يصبح في جهة اليسار والثوابت في جهة اليمين وكالاتي :

$$2Dx - 4x + Dy - y = e^t$$

$$Dx + 3x + D^2y + y = 0$$

ثالثا : نستخرج العامل المشترك وكالاتي :

$$(2D - 4)x + (D - 1)y = e^t --------5$$

$$(D + 3)x + (D^2 + 1)y = 0 --------6$$

نرتب المعادلات بشكل مصفوفة وكما يلي :

$$\begin{bmatrix} 2D-4 & D-1 \\ D+3 & D^2+1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} e^t \\ 0 \end{Bmatrix}$$

رابعا : نحل هذه المعادلات بطريقة كريمر Grammer والتي تم شرحها في الصف الثاني :

$$X = \frac{D'x}{D'} \qquad y = \frac{D'y}{D'}$$

$$x = \frac{\begin{vmatrix} e^t & D-1 \\ 0 & D^2+1 \end{vmatrix}}{\begin{vmatrix} 2D-4 & D-1 \\ D+3 & D^2+1 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 2D-4 & e^t \\ D+3 & 0 \end{vmatrix}}{\begin{vmatrix} 2D-4 & D-1 \\ D+3 & D^2+1 \end{vmatrix}}$$

$$x = \frac{D^2e^t + e^t}{2D^3 - 4D^2 + 2D - 4 - D^2 - 2D + 3}$$

$$x = \frac{e^t + e^t}{2D^3 - 5D^2 - 1} = \frac{2e^t}{2D^3 - 5D^2 - 1}$$

$$y = \frac{-De^t - 3e^t}{2D^3 - 5D - 1} = \frac{-e^t - 3e^t}{2D^3 - 5D^2 - 1} = \frac{-4e^t}{2D^3 - 5D^2 - 1}$$

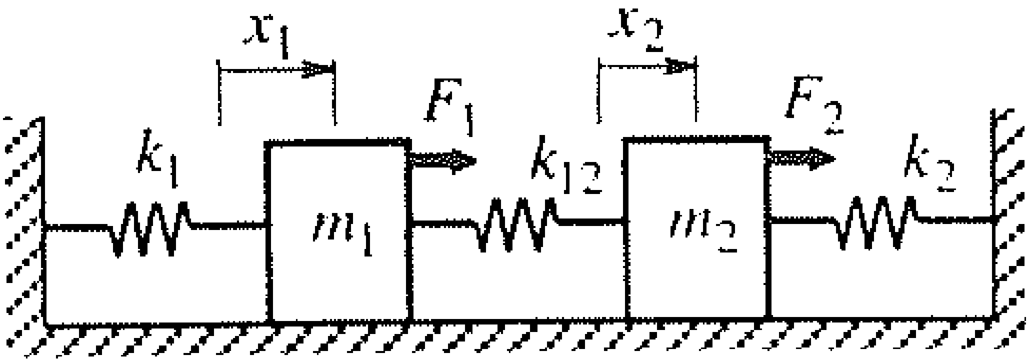
$$(2D^3 - 5D^2 - 1)\ x = 2e^t \ --------------7$$

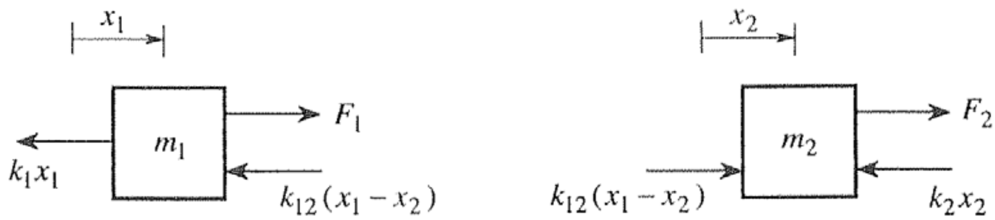
$$(2D^3 - 5D^2 - 1)y = -4e^t \ --------------8$$

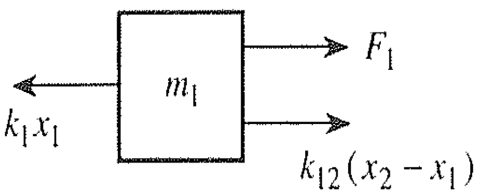
اصبحت المعادلة 7بدلالة x ومعادلة 8 بدلالة y وبحل كل من معادلة 7 ومعادلة 8 ونجد من معادلة 7 كل من xh and xp وكما نجد من معادلة 8 كل من yh, yp.

ومن التطبيقات على المعادلات الخطية التفاضلية هي :

1- spring mass system as shown in following example :







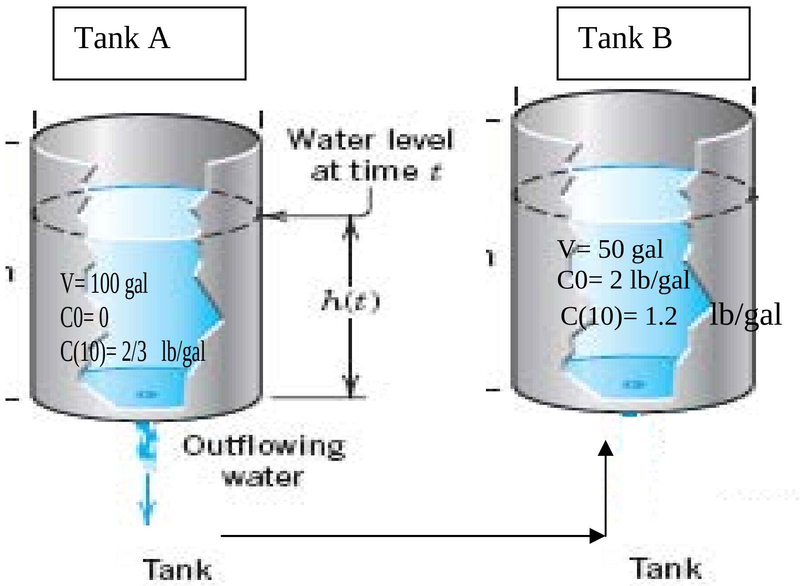
$$m_1x_1'' + (k_1 + k_{12})\,x_1 - k_{12}x_2 = F_1(t),$$
$$m_2x_2'' - k_{12}x_1 + (k_2 + k_{12})\,x_2 = F_2(t).$$

ويتم استخدام الطريقة التي تم ذكرها لحل هذه المعادلات حيث يتم ايجاد كل من x_1 , x_2

2- The amount of the salt in a tank for more than one tank

Example :

Find the amount of salt in tank (A) and tank (B) with respect to time ☹(as a home work).



Even, Odd, and Periodic Functions

Before taking up our study of Fourier series, in the next section, we need to define even, odd, and periodic functions.

Let f be defined on an x interval, finite or infinite, that is centered at $x = 0$. We say that f is an **even** function if

$$\boxed{f(-x) = f(x),} \tag{1}$$

and an **odd** function if

$$\boxed{f(-x) = -f(x),} \tag{2}$$

for all x in that interval. That is, the graph of f is *symmetric* about $x = 0$ if f is even, and *antisymmetric* about $x = 0$ if f is odd. Examples are shown in Fig. 1. For example, $5, x^2, 3x^4, \cos x, \sin|x|$, and e^{-x^2} are even, and $x, 3x^3, 2x^5$, $\sin x$, and $x\cos x$ are odd.

There are several useful algebraic properties of even and odd functions, such as the following:

$$\text{even} + \text{even} = \text{even}, \tag{3a}$$
$$\text{even} \times \text{even} = \text{even}, \tag{3b}$$
$$\text{odd} + \text{odd} = \text{odd}, \tag{3c}$$
$$\text{odd} \times \text{odd} = \text{even}, \tag{3d}$$
$$\text{even} \times \text{odd} = \text{odd}. \tag{3e}$$

To prove (3e), for example, let $F(x)$ be even and let $G(x)$ be odd. Then $F(-x)G(-x) = F(x)[-G(x)] = -F(x)G(x)$, in accord with (2).

In addition, two useful integral properties are as follows. If f is even, then

$$\boxed{\int_{-A}^{A} f(x)\,dx = 2\int_0^A f(x)\,dx, \qquad (f \text{ even})} \tag{4a}$$

and if f is odd, then

$$\boxed{\int_{-A}^{A} f(x)\,dx = 0, \qquad (f \text{ odd})} \tag{4b}$$

for if we interpret the integrals in (4a) as areas (positive above the x axis, negative below it), then the area $\int_{-A}^{0} f(x)\,dx$ is equal to the $\int_0^A f(x)\,dx$ due to the symmetry of the graph of f. And in the case of (4b) the areas $\int_{-A}^{0} f(x)\,dx$ and $\int_0^A f(x)\,dx$ are negatives of each other, due to the antisymmetry of the graph of f, and hence cancel.

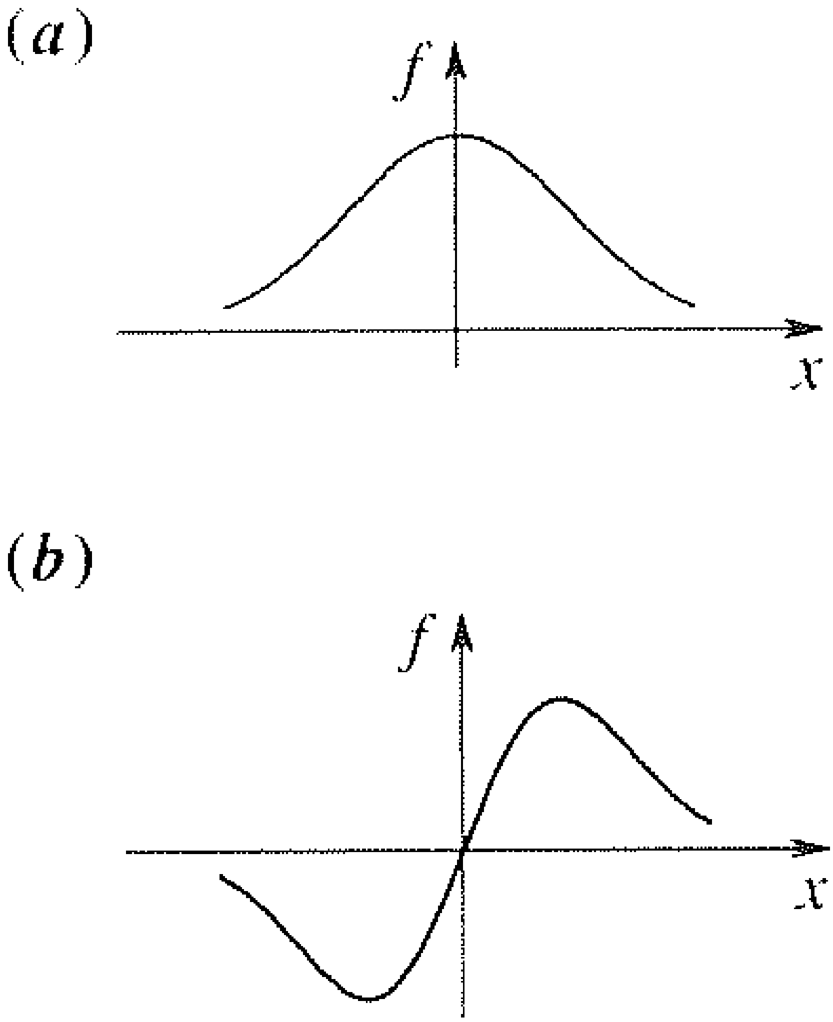


Figure 1. Even and odd.

Alternatively, (4a) and (4b) follow directly from (1) and (2), respectively. For xample, if f is odd, then

$$\begin{aligned}
\int_{-A}^{A} f(x)\,dx &= \int_{-A}^{0} f(x)\,dx + \int_{0}^{A} f(x)\,dx \\
&= \int_{A}^{0} f(-t)\,(-dt) + \int_{0}^{A} f(x)\,dx \qquad (x = -t) \\
&= \int_{0}^{A} f(-t)\,dt + \int_{0}^{A} f(x)\,dx \\
&= \int_{0}^{A} -f(t)\,dt + \int_{0}^{A} f(x)\,dx \qquad (\text{oddness of } f) \\
&= -\int_{0}^{A} f(x)\,dx + \int_{0}^{A} f(x)\,dx \qquad (t = x) \\
&= 0, \qquad (5)
\end{aligned}$$

s stated in (4b).

Note carefully that a given function is not necessarily even or odd; it may be *oth* even and odd, or it may be *neither*. Every function can be uniquely decom-osed into the sum of an even function, say f_e, and an odd function, say f_o, as emonstrated by the simple identity

$$\boxed{\begin{aligned} f(x) &= \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} \\ &\equiv \quad f_e(x) \quad + \quad f_o(x), \end{aligned}} \qquad (6)$$

or observe that

$$f_o(-x) = \frac{f(-x)-f(x)}{2} = -\frac{f(x)-f(-x)}{2} = -f_o(x)$$

nd, similarly, that $f_e(-x) = f_e(x)$.

EXAMPLE 2. For example, $\sin x$ is periodic with period 2π because $\sin(x + 2\pi) = \sin x \cos 2\pi + \sin 2\pi \cos x = \sin x$ for all x. ▮

In graphical terms, one can think of the graph of a periodic function f as generated by stamping it out one period at a time, as with an inked woodblock.

EXAMPLE 3. The function f shown in Fig. 3 is seen to be periodic with period $T = 4$, for if the segment BCD, for instance, is "stamped out" indefinitely to the right and left we generate the graph of f. There is nothing special about choosing the segment BCD for this purpose; ABC, or any other segment of length 4, would do as well. ▮

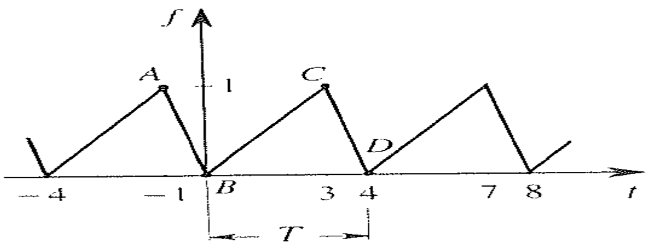


Figure 3. Periodic function f.

Specifically, if $f(x)$ is periodic, of period 2ℓ, then we define the **Fourier series of f**, say FS f, as

$$\text{FS } f = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right),$$

where the coefficients are given by the **Euler formulas**

$$\begin{aligned} a_0 &= \frac{1}{2\ell} \int_{-\ell}^{\ell} f(x)\, dx, \\ a_n &= \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell}\, dx, \qquad n = 1, 2, \ldots \\ b_n &= \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell}\, dx, \qquad n = 1, 2, \ldots \end{aligned}$$

and are known as the **Fourier coefficients** of f.

EXAMPLE 1. *Square Wave.* Consider the "square wave" f shown in Fig. 1, where $f(x)$ is defined as 2 at $x = 0, \pm\pi, \pm 2\pi, \ldots$, as indicated by the heavy dots. Since the period referred to in Theorem 17.3.1 and in (5) is 2ℓ, and the period is seen from Fig. 1 to be 2π, it follows that $\ell = \pi$. Both f and f' are piecewise continuous on $[-\pi, \pi]$, so the theorem applies. Let us use (5) to work out the Fourier series of f and examine its convergence to the square wave f using computer plots of the partial sums of the series.

First, (5b) gives

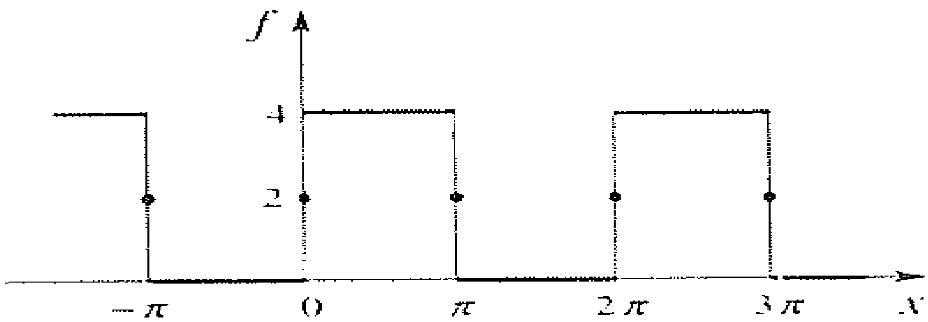


Figure 1. Square wave.

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 4\,dx\right] = 2.$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} 4\,\cos nx\,dx\right] = \left.\frac{4\sin nx}{n\pi}\right|_{0}^{\pi} = 0 \tag{8}$$

Thus the Fourier series representation of f is

$$f(x) = 2 + \frac{8}{\pi}\sum_{1,3,\ldots}^{\infty}\frac{1}{n}\sin nx, \tag{11}$$

where "$1, 3, \ldots$" tells us to omit the terms corresponding to $n = 2, 4, \ldots$. If preferred, (11) can be expressed, equivalently, as

$$f(x) = 2 + \frac{8}{\pi}\sum_{1}^{\infty}\frac{\sin(2n-1)x}{2n-1}, \tag{12}$$

$$f(x) = 2 + \frac{8}{\pi}\sin x + \frac{8}{3\pi}\sin 3x + \frac{8}{5\pi}\sin 5x + \cdots, \tag{13}$$

we define the partial sums of the series as

$$s_1(x) = 2,$$
$$s_2(x) = 2 + \frac{8}{\pi}\sin x,$$
$$s_3(x) = 2 + \frac{8}{\pi}\sin x + \frac{8}{3\pi}\sin 3x,$$

EXAMPLE 2. Let us work out the Fourier series of the periodic function f, the graph of which is given in Fig. 6. Its period is 16, so $\ell = 8$. We can compute a_0 by (5b), but we can see from the figure (dividing the net area over one period by 16) that the average height is $-\frac{1}{4}$ so $a_0 = -\frac{1}{4}$. Further, we see that f is even, in which case (14) gives $b_n = 0$ and

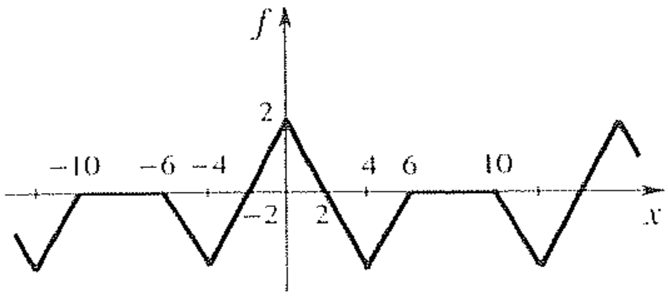


Figure 6. f in Example 2.

$$\begin{aligned}
a_n &= \frac{2}{\ell}\int_0^\ell f(x)\cos\frac{n\pi x}{\ell}\,dx = \frac{2}{8}\int_0^8 f(x)\cos\frac{n\pi x}{8}\,dx \\
&= \frac{1}{4}\left[\int_0^4 (2-x)\cos\frac{n\pi x}{8}\,dx + \int_4^6 (x-6)\cos\frac{n\pi x}{8}\,dx + \int_6^8 0\cos\frac{n\pi x}{8}\,dx\right] \\
&= \frac{1}{4}\left[2\frac{\sin(n\pi x/8)}{n\pi/8} - \frac{1}{(n\pi/8)^2}\left(\cos\frac{n\pi x}{8} + \frac{n\pi x}{8}\sin\frac{n\pi x}{8}\right)\right]\Bigg|_0^4 \\
&\quad + \frac{1}{4}\left[\frac{1}{(n\pi/8)^2}\left(\cos\frac{n\pi x}{8} + \frac{n\pi x}{8}\sin\frac{n\pi x}{8}\right) - 6\frac{\sin(n\pi x/8)}{n\pi/8}\right]\Bigg|_4^6 \\
&= \frac{16}{n^2\pi^2}\left(1 - 2\cos\frac{n\pi}{2} + \cos\frac{3n\pi}{4}\right).
\end{aligned} \tag{18}$$

Thus,

$$f(x) = -\frac{1}{4} + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1 - 2\cos\frac{n\pi}{2} + \cos\frac{3n\pi}{4}}{n^2}\cos\frac{n\pi x}{8}, \tag{19}$$

Note :

$$\int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell}\,\cos\frac{n\pi x}{\ell}\,dx = \begin{cases} 0 & m \neq n \\ \ell & m = n \neq 0 \\ 2\ell & m = n = 0 \end{cases}$$

$$\int_{-\ell}^{\ell} \sin\frac{m\pi x}{\ell}\,\sin\frac{n\pi x}{\ell}\,dx = \begin{cases} 0 & m \neq n \\ \ell & m = n \neq 0 \end{cases}$$

$$\int_{-\ell}^{\ell} \cos\frac{m\pi x}{\ell}\,\sin\frac{n\pi x}{\ell}\,dx = 0 \qquad \text{for all } m, n,$$

where m and n are integers.

Application:

EXAMPLE 3. *Periodically Driven Oscillator.* Consider the driven mechanical oscillator shown in Fig. 7 and governed by the differential equation of motion

$$mx'' + cx' + kx = F(t), \tag{30}$$

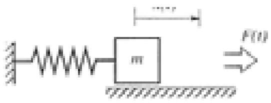


Figure 7. Forced mechanical oscillator.

where m, c, k are the mass, damping coefficient (associated with some combination of viscous damping due to a film of lubricating oil between the mass and the table, and air resistance), and spring stiffness, respectively. Let $m = 1$ kg, $c = 0.04$ kg/sec, and $k = 15$ kg/sec^2, and let $F(t)$ (in newtons) be as shown in Fig. 8. F consists of an endless sequence of pulses, each having unit area (except the first, which is only a "half pulse"). In mechanics, $\int_{t_1}^{t_2} F(t)\,dt$ is the **impulse** delivered by the force F between times t_1 and t_2, so F consists of a periodic sequence of unit impulses, of period 2π. Thus, $\ell = \pi$ in (5). Even though the starting time is $t = 0$, so $t \geq 0$, we can think of F as the extended function shown in Fig. 9, which is even. Thus, if we expand F in a Fourier series we have, for its coefficients, a_0 = average value $= 1/(2\pi)$, $b_n = 0$ because F is even, and, from (14),

$$a_n = \frac{2}{\pi}\int_0^{\pi} F(t)\cos nt\,dt = \frac{2}{\pi}\int_0^{a} \frac{1}{2a}\cos nt\,dt = \frac{\sin na}{n\pi a}. \tag{31}$$

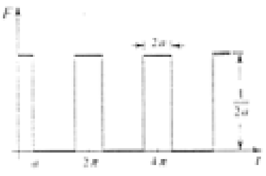


Figure 8. Forcing function F.

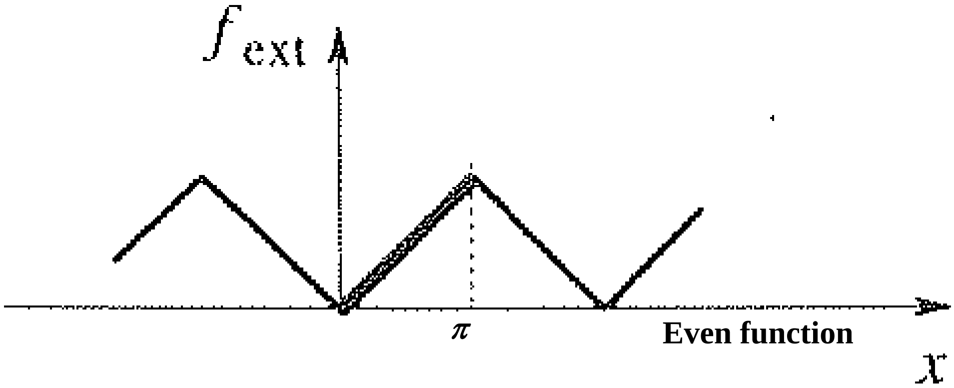
Thus, (30) becomes

$$x'' + 0.04x' + 15x = \frac{1}{2\pi} + \sum_{1}^{\infty} \frac{\sin na}{n\pi a}\cos nt. \tag{32}$$

Half range series:

Some time a function of period 2π is defined over the range 0 to 2π, instead $-\pi$ *to* π, or to 2π. We have a choice to proceed.

1- Even function



For Even function

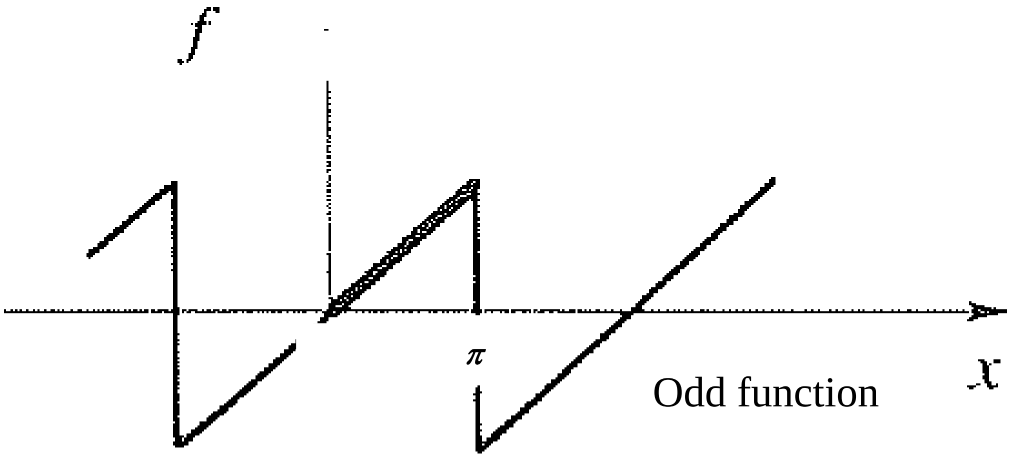
$$f(x) = \sum_{n=1,3,\ldots}^{\infty} a_n \cos\frac{n\pi x}{2L}, \qquad (0 < x < L)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{2L}\,dx,$$

For odd function:

$$f(x) = \sum_{n=1,3,\ldots}^{\infty} b_n \sin \frac{n\pi x}{2L}, \qquad (0 < x < L)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{2L} \, dx.$$



EXAMPLE 1. To illustrate, let us expand the function f, displayed in Fig. 4, in half- and quarter-range cosine and sine series.

HRC: From (3),

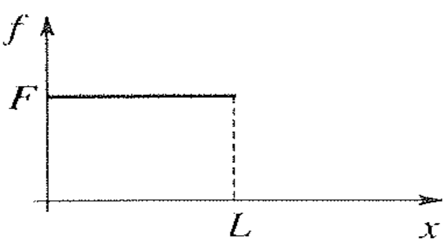
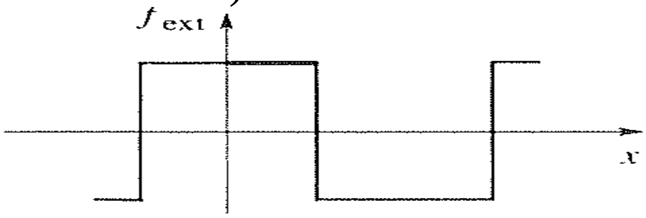


Figure 4. f in Example 1.

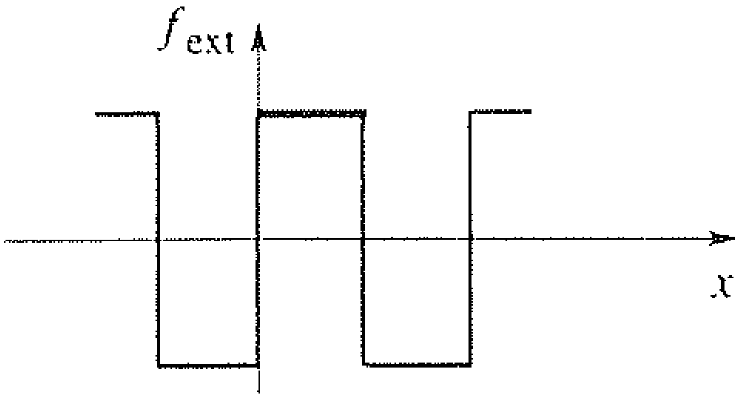
For half range cosine expansion(will be even so bn =0and we have to find a0 and an)



$$a_0 = \frac{1}{L}\int_0^L F\,dx = \frac{FL}{L} = L,$$

$$a_n = \frac{2}{L}\int_0^L F\cos\frac{n\pi x}{L}\,dx = \frac{2F}{n\pi}\sin\frac{n\pi x}{L}\Big|_{x=0}^{x=L} = 0,$$

For half range sine expansion (will be odd function and a0 and an =0 we have to find bn)



$$b_n = \frac{2}{L}\int_0^L F\sin\frac{n\pi x}{L}\,dx = -\frac{2F}{n\pi}(\cos n\pi - 1),$$

$$f(x) = \frac{2F}{\pi}\sum_{n=1}^{\infty}\frac{1 - cos n\pi}{n}\sin\frac{n\pi x}{L},$$

partial differential equation

if it contains partial derivatives of the dependent variable with respect to two or more independent variables.

1. The **diffusion equation**

$$\alpha^2 u_{xx} = u_t \qquad (\alpha^2 = \text{constant})$$

2. The **wave equation**

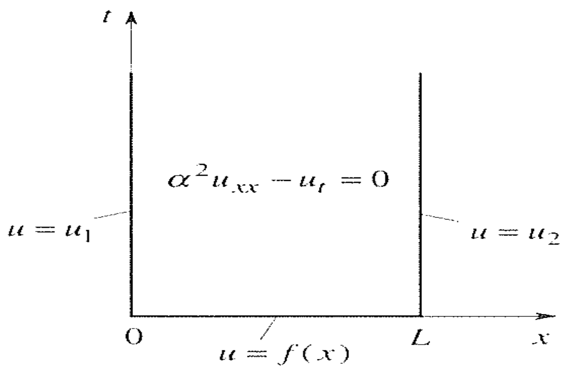
$$c^2 u_{xx} = u_{tt} \qquad (c^2 = \text{constant})$$

3. The **Laplace equation**

$$u_{xx} + u_{yy} = 0$$

EXAMPLE 1. Consider the diffusion problem

$$L[u] = \alpha^2 u_{xx} - u_t = 0, \qquad (0 < x < L,\ 0 < t < \infty)$$
$$u(0,t) = u_1, \quad u(L,t) = u_2, \qquad (0 < t < \infty)$$
$$u(x,0) = f(x), \qquad (0 < x < L)$$



According to the method of **separation of variables** we begin by seeking solutions of

$$\boxed{u(x,t) = X(x)T(t).}$$

Substitute the above equation in the original equation

$$\alpha^2 X''T = XT',$$

Divide both side by XT, we get

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T}.$$

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = \text{constant} = -\kappa^2,$$

This mean that both sides equal constant $-\kappa^2$,

$$\frac{X''}{X} = -\kappa^2 \qquad \text{and} \qquad \frac{1}{\alpha^2}\frac{T'}{T} = -\kappa^2,$$

$$X'' + \kappa^2 X = 0,$$
$$T' + \kappa^2\alpha^2 T = 0.$$

Solving the above two equations will gives the following equations:

$$X = A\cos\kappa x + B\sin\kappa x,$$
$$T = Ce^{-\kappa^2\alpha^2 t}.$$

$$u = XT = (A\cos\kappa x + B\sin\kappa x)Fe^{-\kappa^2\alpha^2 t}$$

To find A, B and C use the boundary condition;
u(0,t) = u1 and u(l,t)=u2

$$F(x) = \sum_{n=1}^{\infty} K_n \sin \frac{n\pi x}{L} \qquad (0 < x < L)$$

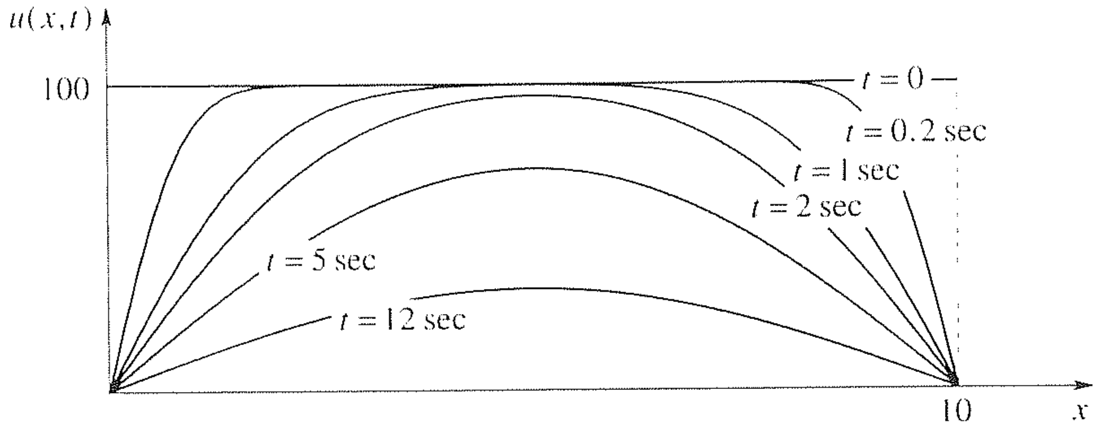
$$K_n = \frac{2}{L} \int_0^L F(x) \sin \frac{n\pi x}{L}\, dx.$$

To illustrate, consider a 10 cm-long copper rod held in boiling water until its temperature is 100° C throughout. At time $t = 0$ it is removed and its ends are quenched with ice for all $t > 0$. (We can either neglect heat loss from its lateral surface or specify that that surface be insulated.) Then $\alpha^2 = 1.14\,\text{cm}^2/\text{sec}$ (for copper), $L = 10\,\text{cm}$, $u_1 = u_2 = 0$, and $F(x) = 100°$ C,

$$K_n = \frac{2}{10} \int_0^{10} 100 \sin \frac{n\pi x}{10}\, dx = \begin{cases} \dfrac{400}{n\pi}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$u(x,t) = \frac{400}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{10}\, e^{-0.1125 n^2 t}.$$

For different time



$u(x,t)$ for the case where $\alpha^2 = 1.14$, $L = 10$, $u_1 = u_2 = 0$, and $f(x) = 100$.

The heat will change at different time steps and as example , different time steps were choice :t= 0.2 , 1 and 12

$$u(x,0.2) = 124.5\sin\frac{\pi x}{10} + 34.7\sin\frac{3\pi x}{10} + 14.5\sin\frac{5\pi x}{10} + \cdots,$$
$$u(x,1) = 113.8\sin\frac{\pi x}{10} + 15.4\sin\frac{3\pi x}{10} + 1.5\sin\frac{5\pi x}{10} + \cdots,$$
$$u(x,12) = 33.0\sin\frac{\pi x}{10} + 0.0002\sin\frac{3\pi x}{10} + 6\times 10^{-14}\sin\frac{5\pi x}{10} + \cdots,$$

التحليلات الهندسية والطرق العددية/ هندسة البناء والانشاءات

توصيف المفردات

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ت	توصيف المادة
1	تطبيقات هندسية للمعادلات التفاضلية الاعتيادية من الدرجة الاولى والثانية : تركيز الاملاح في الخزانات ، التصريف في الخزانات، الاهتزازات الميكانيكية، قوانين نيوتن للحركة
2	المعادلات التفاضلية الانية : حل المعادلات بطريقة كرايمر ، تطبيقات هندسية
3	المعادلات التفاضلية من الدرجة الثانية الخالية من الثوابت ، حل المعادلات بطريقة اويلر ، حل المعادلات بطريقة المتسلسلة الاسية (طريقة فروبينوس).
4	متسلسلة فورير : الدوال الدورية، معاملات فورير ، الدوال الزوجية والفردية ، مفكوك نصف المدى
5	المعادلات التفاضلية الجزئية : حل المعادلات بطريقة فصل المتغيرات، تطبيقات هندسية حول معادلة الموجة ، معادلة التوزيع الحراري
6	المصفوفات : حل المعادلات الاعتيادية الانية بطريقة (matrix inversion) بطريقة الحذف كاوس بطريقة كاوس –جوردن وبطريقة كاوس سيدل
7	مقدمة في الطرق العددية ، جدول الفروقات ، الفروقات المقسمة ، طريقة لاكرنج
8	التداخل العددي طريقة نيوتن كريكوري ، طريقة نيوتن ، طريقة لاكرنج
9	التكامل العددي : طريقة شبه المنحرف وطريقة سمبسون ، طريقة Gaussian quadrate
10	حل المعادلات غير خطية: طريقة نيوتن _رابسون ، طريقة المعاملات الغير محددة ، طريقة الاوزان الغير محددة
11	حل المسائل ذات القيم الابتدائية : طريقة تايلر ، طريقة اويلر ، طريقة اويلر المحدثة ، طريقة رانك _كوتا
12	طريقة الفروقات المحددة لحل مسائل القيم التخمينية